`Dynamical' representation of the Poincaré algebra for higher-spin fields in interaction with plane waves

1999 J. Phys. A: Math. Gen. 32 2499

(http://iopscience.iop.org/0305-4470/32/12/020)

J. Phys. A: Math. Gen. **32** (1999) 2499–2508. Printed in the UK PII: S0305-4470(99)00515-6

'Dynamical' representation of the Poincaré algebra for higher-spin fields in interaction with plane waves

R Saar†‖, R K Loide‡¶, I Ots§+ and R Tammelo†‖

† Institute of Theoretical Physics, Tartu University, Tähe Street 4, Tartu 51010, Estonia
‡ Department of Physics, Tallinn Technical University, Ehitajate Road 5, Tallinn 19086, Estonia
§ Institute of Physics, Tartu University, Riia Street 142, Tartu 51014, Estonia

Received 22 December 1998

Abstract. To avoid the defects of higher-spin interaction theory, the field-dependent invariant representation (the 'dynamical' representation) of the Poincaré algebra is considered as a dynamical principle. A general 'dynamical' representation for a single elementary particle of arbitrary spin in the presence of a plane-wave field is constructed and the corresponding forms of the higher-spin interaction terms found. The properties of relativistically invariant first-order higher-spin equations with the 'dynamical' interaction are examined. It is shown that the Rarita–Schwinger spin-$\frac{3}{2}$ equation with the 'dynamical' interaction is causal and free from algebraic inconsistencies. As distinct from the first-order higher-spin relativistic equations with the minimal coupling, there exist the Klein–Gordon divisors for the first-order equations with the non-minimal, 'dynamical' interaction, and the corresponding Klein–Gordon equations are causal.

1. Introduction

The description of higher-spin particles in interaction is beset with difficulties. It was in the 1960s that defects were found in higher-spin interaction theories. On the quantum level it was demonstrated [1, 2] that in the case of minimal electromagnetic coupling some of the anticommutation relations would become indefinite. It appears that the defects are also present on the classical level. It was revealed that in external electromagnetic fields there appear acausal modes of propagation [3]. Besides, there exist algebraic inconsistencies in some higher-spin interaction theories.

Since the 1960s much work has been done to solve the problems, but no satisfactory results have been obtained by using minimal electromagnetic coupling. The task of finding the origin of the defects of higher-spin interaction theory is still topical.

The search for a consistent higher-spin theory is faced with different difficulties. First, the theory of the relativistic wave equation is based on the representations of the Poincaré group. However, the representations of the Poincaré group in field theory are somewhat specific in their mathematical realization. Secondly, the theory of higher-spin fields is rather complicated and the wave equations and Lagrangians used there are not always correct. Thus it is difficult to say whether the problems connected with higher-spin theories are technical or pertain to principle.

‖ E-mail address: tammelo@physic.ut.ee
¶ E-mail address: karl@edu.ttu.ee
+ E-mail address: ots@fi.tartu.ee

In relativistic particle theory the Poincaré group plays a fundamental role. However, in the case of minimal coupling of the electromagnetic field, in higher-spin theories the Poincaré invariance is violated. One can make a hypothesis that the defects of the higher spin are due to the Poincaré non-invariant minimal coupling, which means that a new dynamical principle is needed. This principle would give minimal coupling in the lower-spin cases ($s = 0, \frac{1}{2}$) and a new, non-minimal Poincaré-invariant interaction in the higher-spin cases.

In this paper, an attempt is made to build a consistent higher-spin theory by using the 'dynamical' representation of the Poincaré algebra as a dynamical principle. This is in agreement with the requirements referred to above. The 'dynamical' representation of the Poincaré algebra was first introduced by Chakrabarti [4] and further studied by Beers and Nickle [5] in the case of spin-$\frac{1}{2}$ particles. We generalize their work by constructing the 'dynamical' representations for arbitrary spins in interaction with the same special external field as used by the above authors. That is, the representations are built by way of introducing a plane electromagnetic field into the free Poincaré algebra. The new 'dynamical' representations are constructed from the generators of the free Poincaré algebra and the external field in such a way that the new, field-dependent generators obey the commutation relations of the free Poincaré algebra. Now, analogously to the free-particle theory, the wave equations with respect to the 'dynamical' representation of the Poincaré algebra can be constructed. So, our principal idea will not be to introduce any auxiliary fields (e.g. supergravity) and particles (e.g. lower spins), but to construct a consistent theory of interacting higher-spin fields by means of only a single particle and external field.

In such a theory, in spite of the presence of the external field, the particle behaves like a free particle. Since the free higher-spin theory has no defects, there is a hope that some of the troubles existing in the minimal coupling theory can be avoided in the 'dynamical' interaction theory. The main goal of the present paper is to examine whether this is really the case.

The paper is organized as follows. In section 2 we construct the 'dynamical' representation of the Poincaré algebra for arbitrary spins and an external plane-wave field. In section 3 the first-order relativistic wave equations for arbitrary spins in the 'dynamical' representation are given. In section 4 the 'dynamical' representation of the Rarita–Schwinger spin-$\frac{3}{2}$ equation, i.e. of the equation with subsidiary conditions, is presented. In section 5 we demonstrate that the 'dynamical' representation of the Rarita–Schwinger spin-$\frac{3}{2}$ equation is causal. Some final remarks are added in section 6.

2. 'Dynamical' representation of the Poincaré algebra for any spin

Let us proceed from commutation relations of the ten-dimensional Lie algebra, $\mathfrak{p}_{1,3}$, of the Poincaré group $P_{1,3} = T_{1,3} \odot SO_{1,3}$ ($T_{1,3}$ denotes the Abelian group of 4-translations and $SO_{1,3}$ is the restricted Lorentz group)

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] &= \mathrm{i}(g_{\mu\sigma} M_{\nu\rho} + g_{\nu\rho} M_{\mu\sigma} - g_{\mu\rho} M_{\nu\sigma} - g_{\nu\sigma} M_{\mu\rho}) \\ [M_{\mu\nu}, P_\sigma] &= \mathrm{i}(g_{\nu\sigma} P_\mu - g_{\mu\sigma} P_\nu) \\ [P_\mu, P_\nu] &= 0. \end{aligned} \tag{2.1}$$

Here the four generators P_μ correspond to the subgroup of translations, and the other six, $M_{\mu\nu}$, to the restricted Lorentz group. The generators have the form:

$$\begin{aligned} P_\mu &= \mathrm{i}\partial_\mu \\ M_{\mu\nu} &= L_{\mu\nu} + S_{\mu\nu} \\ L_{\mu\nu} &= x_\mu P_\nu - x_\nu P_\mu \end{aligned} \tag{2.2}$$

where $S_{\mu\nu}$ are the generators of the finite-dimensional representation of the Lorentz group. The Casimir operators are

$$P^2 = P_\mu P^\mu \qquad W^2 = W_\mu W^\mu \tag{2.3}$$

where $W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^\sigma$ is the common Pauli–Lubanski pseudovector.

Our idea is to introduce an external field invariantly into the Poincaré algebra, i.e. one has to transform the Poincaré generators to be dependent on the external field in such a way that the new, field-dependent generators would obey the commutation relations (2.1). Such a Lie algebra of the Poincaré group is called 'dynamical'. As was shown by Chakrabarti, the simplest way to build the 'dynamical' representation is to introduce the external field by a non-singular transformation

$$U: \mathfrak{p}_{1,3} \to \mathfrak{p}^d_{1,3} = U\mathfrak{p}_{1,3}U^{-1} = \mathfrak{p}_{1,3} + [U, \mathfrak{p}_{1,3}]U^{-1} \tag{2.4}$$

or explicitly

$$\begin{aligned} \pi_\mu &= P_\mu + [U, P_\mu]U^{-1} & \chi_\mu &= x_\mu + [U, x_\mu]U^{-1} \\ \sigma_{\mu\nu} &= S_{\mu\nu} + [U, S_{\mu\nu}]U^{-1} & \lambda_{\mu\nu} &= \chi_\mu \pi_\nu - \chi_\nu \pi_\mu + \sigma_{\mu\nu}. \end{aligned} \tag{2.5}$$

Now the question is, how to find the transformation operator. Does it exist for any field $A_\mu(x)$ or only for the special forms of $A_\mu(x)$? There is no direct prescription for how to find the operator U in general. However, it can be shown [6, 7] that such an operator can be found for a special field. Let the external field be an arbitrary function of $\xi = k \cdot x$

$$A_\mu(x) = A_\mu(\xi) \tag{2.6}$$

with the Lorentz gauge

$$k^\mu A'_\mu = 0 \tag{2.7}$$

where $A'_\mu = (\mathrm{d}A_\mu/\mathrm{d}\xi)$. (Physically speaking, we make a reasonable idealization, describing a laser beam with a plane-wave field characterized by a null vector k_μ.)

The Poincaré algebra $\mathfrak{p}_{1,3}$ and the external field $A_\mu(\xi)$ together generate the 'dynamical' representation $\mathfrak{p}^d_{1,3}$ with the operators

$$\begin{aligned} &P_\mu \to \pi_\mu = P_\mu + k_\mu h' + k_\mu b F_{\sigma\rho} S^{\sigma\rho} \\ &S_{\mu\nu} \to \sigma_{\mu\nu} = S_{\mu\nu} + 2b(g_{\mu\rho}G_{\nu\sigma} - g_{\nu\rho}G_{\mu\sigma} + bg_{\mu\rho}(G)^2_{\nu\sigma} - bg_{\nu\rho}(G^2)_{\mu\sigma} + bG_{\mu\nu}G_{\rho\sigma})S^{\rho\sigma} \\ &x_\mu \to \chi_\mu = x_\mu + \mathrm{i}[h + bG_{\sigma\rho}S^{\rho\sigma}, x_\mu] \\ &L_{\mu\nu} \to \lambda_{\mu\nu} = \chi_\mu \pi_\nu - \chi_\nu \pi_\mu \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} &h' = \frac{\mathrm{d}h}{\mathrm{d}\xi} \qquad b = \frac{-e}{2K_p} \qquad bG_{\mu\nu} = k_\mu f_\nu - k_\nu f_\mu \\ &F_{\mu\nu} = G'_{\mu\nu} \qquad (G^2)_{\mu\nu} = G_{\mu\rho}G^\rho_\nu. \end{aligned} \tag{2.9}$$

The parameter e is the charge of the particle described by the wavefunction $\psi(x)$.

The operator $K_p \equiv k_\mu P^\mu$ commutes with any other and plays a special role in the theory. Further it is assumed that its inverse, $1/K_p$, exists in case it is needed for the construction of the theory, i.e. the operator $1/K_p$ is assumed to be non-singular.

According to the line of thought presented by relation (2.4), realization of (2.8) can be achieved by the non-singular transformation

$$U = \exp \mathrm{i}(h + bG_{\sigma\rho}S^{\sigma\rho}) \tag{2.10}$$

and, therefore, the construction of the 'dynamical' representation in the case of the field on which the restrictions (2.6) and (2.7) are valid is fulfilled. With the exception of the special form of the field the constructed 'dynamical' representation is the most general representation for an arbitrary spin.

To use the representation reasonably it needs a specification. Having introduced the external field into the Poincaré algebra, we have introduced a certain form of interaction into the theory. It is reasonable to specify the representation in such a way that, in the spin-0 and spin-$\frac{1}{2}$ cases, the interaction introduced with the help of the 'dynamical' representation would coincide with the minimal electromagnetic coupling. The desired result can be achieved by choosing

$$f_\mu = \frac{-e}{2K_p}A_\mu \qquad h' = \frac{e}{K_p}\left(\tfrac{1}{2}eA^2 - AP\right).$$

From equations (2.9)

$$G_{\mu\nu} = k_\mu A_\nu - k_\nu A_\mu \qquad F_{\mu\nu} = k_\mu A'_\nu - k_\nu A'_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$$

and the generators of the 'dynamical' Poincaré algebra $\mathfrak{p}^d_{1,3}$ have the form:

$$\begin{aligned}\pi_\mu &= P_\mu + k_\mu \frac{e}{2K_p}(eA^2 - 2A\cdot P - F_{\sigma\rho}S^{\sigma\rho})\\ \sigma_{\mu\nu} &= S_{\mu\nu} - \frac{e}{K_p}\bigg(\frac{e}{2K_p}A^2(g_{\mu\rho}k_\nu - g_{\nu\rho}k_\mu)k_\sigma + g_{\mu\rho}(k_\nu A_\sigma - k_\sigma A_\nu)\\ &\qquad -g_{\nu\rho}(k_\mu A_\sigma - k_\sigma A_\mu) - \frac{e}{K_p}(k_\mu A_\nu - k_\nu A_\mu)k_\rho A_\sigma\bigg)S^{\rho\sigma}\\ \chi_\mu &= x_\mu - \frac{e}{2K_p}\left[x_\mu, \int \mathrm{d}\xi\ (eA^2 - 2A\cdot P) - G_{\sigma\rho}S^{\sigma\rho}\right].\end{aligned} \tag{2.11}$$

These generators can be found by applying to the free Poincaré generators the operator

$$U = U_0 \cdot U(s) \tag{2.12}$$

where

$$U_0 = \exp\left[\mathrm{i}\int \frac{\mathrm{d}\xi}{2K_p}[2eP\cdot A(\xi) - e^2A^2(\xi)]\right] \tag{2.13}$$

and

$$U(s) = \exp\left[-\mathrm{i}\frac{e}{2K_p}(k_\mu A_\nu - k_\nu A_\mu)S^{\mu\nu}\right]. \tag{2.14}$$

3. 'Dynamical' interactions

We have shown that in the case of a special external electromagnetic field there exists the operator U which transforms the free Poincaré algebra $\mathfrak{p}_{1,3}$ into the 'dynamical' representation $\mathfrak{p}^d_{1,3}$. By analogy with the free-particle case one can realize the 'dynamical' representation $\mathfrak{p}^d_{1,3}$ on the solution space of the relativistically invariant equations. If in the 'dynamical' representation of equations we write the operators explicitly in terms of the free-field operators, as in equation (2.11), we can find the forms of the Poincaré-invariant 'dynamical' interactions, which will be demonstrated below. Since any system of higher-order differential equations can be reduced to a first-order system, let us start with a first-order relativistic system

$$(P^\mu\beta_\mu - m)\Phi = 0. \tag{3.1}$$

The requirement of relativistic invariance imposes the following conditions on the β-matrices:

$$[\beta_\mu, S_{\rho\sigma}] = \mathrm{i}(g_{\mu\rho}\beta_\sigma - g_{\mu\sigma}\beta_\rho). \tag{3.2}$$

The requirement of relativistic invariance also implies the Klein–Gordon condition, which claims that equation (3.1) can be reduced to the Klein–Gordon equation for every component of the multi-component wavefunction Φ by a finite number of differentiations and algebraic operations. Such an operator which transforms the first-order relativistic wave equation into the second-order Klein–Gordon equation is called the Klein–Gordon divisor [8] and the requirement can be written as follows:

$$d_{KG}(P^{\mu}\beta_{\mu} - m) = P^2 - m^2. \tag{3.3}$$

Further, equation (3.1) describes a system of many spins. To convert the equation into that of the single-particle theory one has to impose supplementary conditions on the β-matrices. These conditions depend on the value of the spin of the particle we want the equation to describe.

Applying the operator U to equation (3.1), one finds

$$U : (P^{\mu}\beta_{\mu} - m)\Phi \to (\pi_{\mu}\Gamma^{\mu} - m)\Phi^{d} = 0 \tag{3.4}$$

where

$$\pi_{\mu} = UP_{\mu}U^{-1} \qquad \Gamma^{\mu} = U\beta^{\mu}U^{-1} \qquad \Phi^{d} = U\Phi.$$

The Γ-matrices satisfy the requirement of relativistic invariance with respect to the 'dynamical' representation

$$[\Gamma_{\mu}, \sigma_{\rho\sigma}] = \mathrm{i}(g_{\mu\rho}\Gamma_{\sigma} - g_{\mu\sigma}\Gamma_{\rho}) \tag{3.5}$$

and all the other requirements needed for the construction of a single-particle theory.

In order to obtain the Γ-matrices we make use of the transformation U in its explicit form, equation (2.14). We find

$$\Gamma_{\mu} = V_{\mu\rho}\beta^{\rho} = \beta_{\mu} - \frac{e}{K_p}\left(\frac{e}{2K_p}A^2k_{\mu}k_{\rho} + G_{\mu\rho}\right)\beta^{\rho}$$

where the

$$V_{\mu\rho} = \left(\exp^{-(e/K_p)G}\right)_{\mu\rho}$$

obey the relation

$$V^{\mu}_{\rho}V_{\mu\rho} = g_{\rho\sigma}.$$

Now, using the explicit forms of π_{μ} and Γ_{μ}, we derive the 'dynamical' interaction in the language of free relativistic wave equations as follows:

$$(\pi_{\mu}\Gamma^{\mu} - m)\Phi^{d} = \left(D_{\mu}\beta^{\mu} - \frac{e}{2K_p}\not{k}\not{F} - m\right)\Phi^{d} = 0. \tag{3.6}$$

Here $\not{k} = k_{\mu}\beta^{\mu}$ and $\not{F} = F_{\mu\nu}S^{\mu\nu}$.

It is important to notice that for the 'dynamical' interaction we obtain the following formal substitution:

$$P_{\mu} \to P_{\mu} - eA_{\mu} - \frac{e}{2K_p}k_{\mu}\not{F}$$

which differs from the minimal substitution with respect to the added term $-(e/K_p)k_{\mu}F$, and which does not coincide with the transformation $P_{\mu} \to \pi_{\mu}$. Finally, from equation (3.6) it can be seen that the 'dynamical' interaction coincides with the minimal one if the second term in the equation is equal to zero. This must naturally be the case for spin-0 and spin-$\frac{1}{2}$ particle equations: the operator U has been chosen to be in agreement with the minimal coupling. Demonstration of the above-mentioned facts is straightforward.

The Kemmer–Duffin spin-0 free-particle theory is defined by equation (3.1), where the β-matrices satisfy the relations

$$\beta_\mu\beta_\nu\beta_\sigma + \beta_\sigma\beta_\nu\beta_\mu = g_{\mu\nu}\beta_\sigma + g_{\nu\sigma}\beta_\mu$$

with the supplementary condition

$$\beta_\mu\beta_\nu\beta_\sigma = 0 \qquad \text{if} \quad \mu \neq \nu \neq \sigma \neq \mu$$

and the Lorentz operators are defined as

$$S_{\mu\nu} = \mathrm{i}[\beta_\mu, \beta_\nu].$$

It can easily be shown that from the above properties of the β-matrices it follows that

$$k_\mu F_{\rho\sigma}\beta^\mu S^{\rho\sigma} = 0.$$

In the Dirac spin-$\frac{1}{2}$ particle case $\beta_\mu = \gamma_\mu$ and $S_{\mu\nu} = \mathrm{i}\frac{1}{4}[\gamma_\mu, \gamma_\nu]$. Here it can also be easily verified that

$$k_\mu F_{\rho\sigma}\gamma^\mu S^{\rho\sigma} = 0.$$

However, for spins higher than $\frac{1}{2}$ the second term in equation (3.6) generally differs from zero, which means that the interaction induced by the 'dynamical' representation is non-minimal.

The question is whether the new, non-minimal interaction will enable avoidance of some of the defects of the higher-spin interaction theory.

4. Rarita–Schwinger equation in a 'dynamical' interaction

The free spin-$\frac{3}{2}$ particle Rarita–Schwinger equation [9] is given as

$$(P_\mu\gamma^\mu - m)\psi_\sigma = 0 \tag{4.1a}$$

$$\gamma_\mu\psi^\mu = 0 \tag{4.1b}$$

$$P_\mu\psi^\mu = 0. \tag{4.1c}$$

Usually the first equation in the system is called the true equation of motion and the two others are subsidiary conditions.

Not all the equations in (4.1) are independent. Indeed, by multiplying the equation from the left by γ^σ and by using the first subsidiary condition one obtains the last condition. Multiplying the first equation (4.1) by $\not{P} + m$ one obtains the Klein–Gordon equation

$$(P^2 - m^2)\psi_\mu = 0.$$

Let us now transform equation (4.1) into the 'dynamical' representation. In order to do this one must cast the equation into the matrix form [10],

$$[P_\mu(1 \otimes \gamma^\mu) - m]\psi = 0 \tag{4.2a}$$

$$(E_{\mu\nu} \otimes \gamma^\nu)\psi = 0 \tag{4.2b}$$

$$P^\mu(E_{\rho\mu} \otimes 1)\psi = 0 \tag{4.2c}$$

where

$$(E_{\mu\nu})^\rho_\sigma = g^\rho_\mu g_{\nu\sigma}.$$

Now we can apply the transformation (2.12), in which for spin $\frac{3}{2}$

$$S_{\mu\nu} = -\mathrm{i}e_{\mu\nu} \otimes 1 + 1 \otimes s_{\mu\nu}$$

where $(e_{\mu\nu})^{\rho}_{\sigma} = -g^{\rho}_{\mu}g_{\nu\sigma} + g_{\mu\sigma}g^{\rho}_{\nu}$ and $s_{\mu\nu} = \mathrm{i}\frac{1}{4}[\gamma_\mu, \gamma_\nu]$, so that the matrix U takes the form:

$$U = U_0(U_P \otimes U_D) \tag{4.3a}$$

with

$$U_P = \exp\left[-\frac{e}{2K_p}(k_\mu A_\nu - k_\nu A_\mu)e^{\mu\nu}\right] \tag{4.3b}$$

as the spin-1 (Proca) part and

$$U_D = \exp\left[-\frac{e}{4K_p}(k_\mu A_\nu - k_\nu A_\mu)\gamma^\mu\gamma^\nu\right] \tag{4.3c}$$

as the bispinor (Dirac) part [11].

Applying the operator U defined by equations (4.3) to equation (4.2) one obtains the Rarita–Schwinger equation in the 'dynamical' representation as follows:

$$\left[(\not{D} - m)g_{\mu\nu} - \frac{\mathrm{i}e}{K_p}\ \not{k}F_{\mu\nu}\right]\Psi^{d\nu} = 0 \tag{4.4a}$$

$$\gamma_\mu \Psi^{d\mu} = 0 \tag{4.4b}$$

where $\Psi^d \equiv U\psi$ and $\not{k} \equiv k_\mu\gamma^\mu$.

The first equation is the true equation of motion because it contains all the derivatives $D_\rho\Psi^d_\sigma$. The static constraint (4.4b) is needed in order to eliminate all the superfluous spin-$\frac{1}{2}$ components.

In the same way, using the operator U or applying the operator $(\not{D}+m)$ to the system (4.4) one obtains

$$(P^2 - m^2)\psi_\mu = 0 \rightarrow \left[\left(\not{D}^2 - m^2\right)g_{\mu\rho} - 2\mathrm{i}eF_{\mu\rho}\right]\Psi^{d\rho} = 0 \tag{4.5}$$

$$P_\rho\psi^\rho = 0 \rightarrow \left(D_\rho - \frac{\mathrm{i}e}{4K_p}\ \not{F}k_\rho\right)\Psi^{d\rho} = 0 \tag{4.6}$$

where $\not{F} \equiv F_{\rho\sigma}\gamma^\rho\gamma^\sigma$.

As in the free-particle case only the equations in (4.4) are independent. Contracting the first equation (4.4a) with γ_μ one obtains equation (4.6) and hence the dynamical interaction is algebraically consistent.

5. 'Dynamical' interaction and causality properties

The main imperfection of the higher-spin theories with minimal coupling is the existence of solutions which propagate acausally. The problems associated with acausality of higher-spin equations, and in particular those for spin-$\frac{3}{2}$ fields, have been the subject of many studies [1, 3, 12–16].

The causality properties of the relativistic first-order wave equations can be investigated, on the one hand, from the equations themselves by using the Courant method [17] of wavefronts and, on the other hand, by the Klein–Gordon equation deduced from the first-order equation by applying the Klein–Gordon divisor, if such an operator exists.

The former possibility of examining the causality properties of the equations is quite complicated because, due to the singularity of the β-matrices, the Courant method cannot be applied directly. Nevertheless, we use the Courant method after decomposing the solution

Ψ^d into two terms, one of which is a transverse vector–bispinor field Φ_μ and the other is determined by a bispinor field B, i.e. [12, 13]

$$\Psi^d_\mu = \Phi_\mu + \tfrac{1}{4}\gamma_\mu \not{D} B \tag{5.1}$$

where

$$D_\mu \Phi^\mu = 0. \tag{5.2}$$

Substituting this expansion into equations (4.5) one obtains

$$\not{D}\Phi_\mu - \frac{1}{2} D_\mu \gamma_\rho \Phi^\rho - m\Phi_\mu + \frac{m}{4}\gamma_\mu\gamma_\rho\phi^\rho - \frac{\mathrm{i}e}{K_p}\not{k} F_{\mu\rho}\Phi^\rho$$
$$+ \frac{\mathrm{i}e}{8K_p} k_\mu \not{F}\gamma_\rho\Phi^\rho - \frac{\mathrm{i}e}{4K_p}\gamma_\mu \not{F} k_\rho \Phi^\rho = 0 \tag{5.3a}$$
$$\not{D} B + \gamma_\rho \Phi^\rho = 0. \tag{5.3b}$$

To account for the expansion (5.1) we note that $\{\Phi_\mu, B\}^3_{\mu=0}$ denote the new dynamical variables for which equations (5.3) provide the true equations of motion. Furthermore, the expansion (5.1) presupposes the invariance under the gauge transformation

$$\Phi_\mu \to \Phi_\mu + \gamma_\mu \not{D}\Lambda \tag{5.4}$$
$$B \to B - 4\Lambda$$

where Λ is an arbitrary bispinor solution of the wave equation

$$\not{D}^2 \Lambda = 0. \tag{5.5}$$

Thus the number of independent dynamical variables has been reduced to 16.

To investigate the nature of the propagation one must calculate the characteristic determinant $D(n)$ and examine its characteristic roots n_μ. The equation is hyperbolic if for any unit space vector, $\vec{n}$, $\vec{n}^2 = 1$, all the values of the n_0 solutions of $D(n) = 0$ are real, and the theory is causal if all satisfy $\|n_0\| \leqslant 1$ [17]. To find the characteristic determinant of equation (5.3), we replace D_μ by n_μ in the derivatives and calculate the determinant $\Delta(n)$ of the resulting coefficient matrix:

$$\Delta(n) = \left(\tfrac{1}{2}\right)^4 \left(n^2\right)^8.$$

Therefore, the system (5.3) is obviously hyperbolic, every characteristic surface is the lightcone and the propagation is causal.

To study the causality properties of the first-order wave equations by means of the corresponding Klein–Gordon equation, one must suppose the existence of the Klein–Gordon divisor d_{KG}. In general, the time coefficient matrix β_0 is singular and thereby equation (3.1) is not a true equation of motion: the time derivatives of certain field components are not determined. In the free-field case a true equation of motion may be obtained by using $d_{KG}(P)$, which is a polynomial in P_μ and β_μ, and is easily calculated with the help of the minimal polynomial of β_μ.

For the interaction case the substitution $P_\mu \to D_\mu$ does not provide a divisor $d_{KG}(D)$ which would yield a second-order equation. It has been shown by Cox [18] that for a causal theory based on the wave equation (3.1) interacting minimally with an external field, severe supplementary conditions are needed.

One can demonstrate that, in the case where the β-matrices are such that for equation (3.1) there exists a Klein–Gordon divisor, there also exists the divisor for equation (3.4) (or (3.7)) generated by the transformation

$$U\colon d_{KG} \to \Theta_{KG} = U d_{KG} U^{-1}. \tag{5.6}$$

Application of the 'dynamical' divisor to equation (3.4) gives

$$\Theta_{KG}(\pi_\mu\Gamma^\mu - m)\Phi^d = (\pi^2 - m^2)\Phi^d = (D^2 - eF_{\rho\sigma}S^{\rho\sigma} - m^2)\Phi^d = 0. \tag{5.7}$$

The second-order equation is the arbitrary-spin generalization of the Feynmann–Gell–Mann equation, or the Klein–Gordon equation with 'dynamical' (interaction) coupling. Since the coefficient matrices before the highest (second-order) derivatives are equal to a unit matrix one may conclude that all the equations (3.1) which have the Klein–Gordon divisor describe, in the presence of 'dynamical' interaction, a causal propagation.

Finally, we consider an example of a spin-$\frac{3}{2}$ particle. It has been shown [3, 13] that equation (3.1), in the presence of interaction for spin $\frac{3}{2}$, has solutions which propagate acausally. However, the results of the analysis presented above demonstrate that in the case of 'dynamical' coupling with a plane-wave field A_μ the first-order spin-$\frac{3}{2}$ equation describes causal propagation. Indeed, according to the concept of weak discrete symmetries the most general equation equivalent to that of Rarita–Schwinger (4.1) is [19]

$$\{P_\sigma\gamma^\sigma g_{\mu\rho} + Y_1\gamma_\mu P_\rho + Y_2 P_\mu\gamma_\rho + Y_3 P_\sigma\gamma^\sigma\gamma_\mu\gamma_\rho - mg_{\mu\rho}\}\psi^\rho = 0. \tag{5.8}$$

Here the coefficients $Y_\mu \equiv y_\mu + \bar{y}_\mu\gamma_5$, $y_\mu, \bar{y}_\mu \in R$ obey the nilpotency conditions

$$\begin{aligned}
&Y_0\bar{Y}_0 = 1\\
&1 + Y_1 + Y_2 + Y_3 + 3Y_1(Y_2 + 2Y_3) = 0\\
&2Y_1 + Y_2 - 2Y_3 - \bar{Y}_1 + 2(Y_1 + Y_2 - 2Y_3)\bar{Y}_1 = 0\\
&2 + 3Y_1 + 3(1 + 2Y_1)(\bar{Y}_2 + 2\bar{Y}_3) = 0.
\end{aligned}$$

(Here by definition $\bar{Y}_i = y_i - \bar{y}_i\gamma_5$.)

The Klein–Gordon divisor of equation (5.8) has the following form:

$$d_{KG}(P) = m + (P\beta) + \frac{1}{m}\left[(P\beta)^2 - P^2\right] + \frac{1}{m^2}(P\beta)\left[(P\beta)^2 - P^2\right]$$

where $(P\beta) \equiv P_\mu\beta^\mu$ and

$$(\beta^\sigma)_{\mu\nu} = Y_0 g_{\mu\nu}\gamma^\sigma + Y_1 g^\sigma{}_\nu\gamma_\mu + Y_2 g^\sigma{}_\mu\gamma_\nu + Y_3\gamma^\sigma\gamma_\mu\gamma_\nu. \tag{5.9}$$

In the presence of 'dynamical' interaction one finds

$$\left\{(Y_0\not{D} - m)g_{\mu\nu} - Y_3\gamma_\mu\not{D}\gamma_\nu - \frac{\mathrm{i}e}{K_p}Y_0\not{k}F_{\mu\nu} + Y_1\gamma_\mu L_\nu + (Y_2 + 2Y_3)L_\mu\gamma_\nu\right\}\Psi^{d\mu} = 0 \tag{5.10}$$

where $L_\mu \equiv D_\mu - (\mathrm{i}e/4K_p)k_\mu\not{F}$ and $\not{F} \equiv F_{\rho\sigma}\gamma^\rho\gamma^\sigma$. However, in spite of the equivalence of the free equations (4.1) and (5.8), it is extremely difficult to prove the equivalence of equations (4.4) and (5.10) if the Y_μ satisfy the nilpotency conditions. Nevertheless, the 'dynamical' divisor Θ_{KG} transforms equation (5.10) into equation (4.5) and therefore a pure spin-$\frac{3}{2}$ equation in the presence of 'dynamical' interaction, equation (5.10) describes a causal propagation of waves.

6. Final remarks

The basic advantage of the 'dynamical' representation is the consistent and causal theory of the single spin-$\frac{3}{2}$ particle. Naturally, some problems arise. For instance, establishing the equivalence between the Rarita–Schwinger equation and the Bhabha-type equation in the presence of a 'dynamical' interaction. The technical complications result in the calculation of constraints and therefore, the Velo–Zwanziger method for estimation of causality proves inapplicable. To test whether or not the first-order equation (3.6) suffers from acausality of propagation, one could make use of the shock-wave formalism of Madora and Tait [20], which, however, is the subject of a separate study.

Acknowledgments

This work was partly supported by the Estonian Science Foundation grant no 3458 for which herein the authors extend their gratitude. We also thank Leo Kaagjärv, MSc, for a critical reading of the manuscript and helpful comments.

References

[1] Johanson K and Sudarshan E C G 1961 *Ann. Phys., NY* **13** 126
[2] Schwinger J 1963 *Phys. Rev.* **30** 800
[3] Velo G and Zwanziger D 1969 *Phys. Rev.* **186** 1337
[4] Chakrabarti A 1968 *Nuovo Cimento* A **56** 604
[5] Beers B and Nickle H H 1972 *J. Math. Phys.* **13** 1592
[6] Saar R, Ots I, Loide R-K and Vili R 1997 *Hadronic J.* **20** 559
[7] Saar R, Loide R-K and Ots I 1989 *Trans. Inst. Phys. Estonian Acad. Sci.* **64** 52
[8] Aurilia A and Umezawa H H 1967 *Nuovo Cimento* A **51** 14
[9] Rarita W and Schwinger I 1941 *Phys. Rev.* **60** 61
[10] Loide R-K, Ots I and Saar R 1988 Tensor-type subsidiary conditions in higher-spin field theory *Preprint* F-43 Academy of Sciences of the Estonian SSR, Tartu p 30
[11] Saar R, Ots I and Mättas L 1994 *Proc. Estonian Acad. Sci. Phys. Math.* **43** 96
[12] Velo G and Zwanziger D 1969 *Phys. Rev.* **188** 2218
[13] Chamaly A and Capri A Z 1977 *Ann. Phys., NY* **74** 503
[14] Zañada A F and Rodeo S 1980 *Phys. Rev.* D **22** 385
[15] Prabhakaran J, Seetharaman M and Mathews P M 1975 *J. Phys. A: Math. Gen.* **8** 560
[16] Kobayashi M and Takahashi Y 1987 *J. Phys. A: Math. Gen.* **20** 6581
[17] Courant R 1962 *Partial Differential Equations* (New York)
[18] Cox W 1976 *J. Phys. A: Math. Gen.* **9** 1025
[19] Saar R, Kõiv M, Ots I and Loide R-K 1993 *J. Math. Phys.* **34** 2806
[20] Madora J and Tait W 1973 *Commun. Math. Phys.* **30** 201